



Spectrum Analysis and Fourier Series *or where did all those frequencies come from?*

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Purpose:

To demonstrate FFT capable instrumentation and its use in studying periodic signals.

Equipment:

- Agilent 54600 series oscilloscope with FFT measurement module, or an equivalent spectrum analyzer.
- Optional: Agilent 33120A function generator or equivalent.

Supplementary Readings:

The Benchtop Electronics Reference Manual, by V. Veley, TAB Books, 2nd ed., 1990
HP 54600A and HP 54601A Oscilloscopes, Hewlett-Packard, 1991, Publication # 54600-90901,
HP 54657A, and HP 54658A Measurement/Storage Modules, HP, 1992, Publication # 54657-90904

Theory:



Any single-valued, finite and continuous function $V(t)$, as shown in figure 1, having a period of $2\pi/\omega$, may be expressed as Fourier's Theorem.

Fourier's Theorem: $V(t) = a_0 + a_1 \sin(\omega t + \theta_1) + a_2 \sin(2\omega t + \theta_2) + a_3 \sin(3\omega t + \theta_3) + \dots$
 where: $w = 2\pi$ radians; and $t =$ time in seconds
 But, $A \sin(\omega t + \theta) = A \sin(\omega t) \cos(\theta) + A \cos(\omega t) \sin(\theta) = a \cos(\omega t) + b \sin(\omega t)$
 where: $a = A \sin(\theta)$; and $b = A \cos(\theta)$
 Therefore, $V(t) = a_0 + a_1 \cos(\omega t) + a_2 \cos(2\omega t) + a_3 \cos(3\omega t) + \dots$
 $+ b_1 \sin(\omega t) + b_2 \sin(2\omega t) + b_3 \sin(3\omega t) + \dots$

This means that the waveform is composed of an average value (a_0) together with fundamental sine and cosine waves and their harmonics. For the square wave above we need



to determine the coefficients $a_0, a_1, a_2, \dots, b_1, b_2, b_3, \dots$. This is done by using calculus, which is outlined in Derivation below. Equation 1 is the result.

$$\text{Equation 1: } V(t) = \frac{V_0}{2} + \frac{2V_0}{\pi} \left[\sin(\omega t) + \frac{1}{3} \sin(3\omega t) + \frac{1}{5} \sin(5\omega t) + \dots + \frac{1}{n} \sin(n\omega t) + \dots \right]$$

The coefficient a_0 is the average voltage, and all the other "a" coefficients are zero (no cosine terms) due to the wave's asymmetry around the vertical axis. Table 1 gives the computed value of the first five non-zero coefficients.

Table 1	a_0	b_1	b_3	b_5	b_7	b_9
Theoretical coefficients	$0.5V_0$	$0.637V_0$	$0.212V_0$	$0.127V_0$	$0.091V_0$	$0.091V_0$

Figure 2a shows the fundamental (A) and the third harmonic (B). The composite of these two frequencies (C) starts to approximate the original square wave of figure 1. Figure 2b shows the fifth harmonic (D) and the composite (E) of the fundamental, third, and the fifth harmonic. Finally, figure 2c adds the seventh harmonic (F). The composite (G) of the fundamental, third, fifth, and seventh harmonics makes a closer approximation to the original square wave of figure 1. More harmonics are needed to improve the fit. A symmetrical square wave such as this contains a large number of strong harmonics.

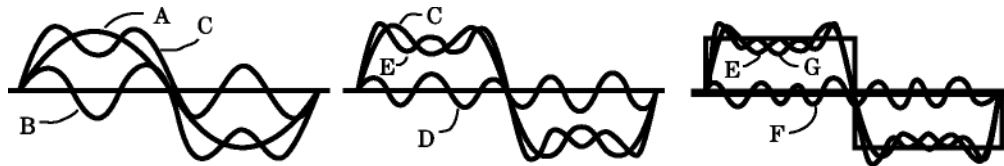


Figure 2a

Figure 2b

Figure 2c

Experimental Procedure:

1. Connect input 1 of the oscilloscope to the Oscilloscope's probe adjustment signal on the front panel (0 to 5V, 1.2 kHz squarewave) or connect to the function generator with the output similarly adjusted.
2. Press Autoscale. Press channel button 1, set softkey Coupling to DC, and set Probe softkey to proper setting (if no probe, set to 1). Press Time, then softkey Freq, and record in Table 2 frequency (f) of square wave. Compute and record the other harmonic frequencies (i.e., 3f, etc.). Press Voltage, press softkeys Vp-p and Vavg. Record Vp-p as amplitude V_0 and the average voltage in Table 2. Using V_0 and Table 1, fill in the theoretical values for the coefficients in Table 2.



Table 2 $V_0 =$ _____					
$V_{avg} =$ _____		f	3f	5f	7f
Harmonic frequencies					
Measured frequencies					
% difference					
	a_0	b_1	b_3	b_5	b_7
Theoretical coefficients					
Harmonic amplitude (dB)					
Experimental coefficient					
% difference					

- Adjust Time/div knob to 1ms/div. Press softkey 1 to Off. Press Display and set Vectors On.
- Set the oscilloscope to the FFT mode: press button +/-; set Function 2 to On; press the softkey Menu; set Operand to 1; set Operation to FFT; set Units/div to 10 dB; set Ref Level to 20 dB; press softkey FFT Menu; set Freq Span to 12 kHz; push softkey Move 0 Hz to Left; and set Window to Hanning (use Hanning to measure frequencies of peaks). Observe the unique amplitude versus frequency display. Print, photograph, or sketch the display.
- Measure the frequencies of the harmonic peaks: press Cursor; press Active Cursor, set f1 to On; use the knob below Cursor to align the movable dashed line to the first peak; record f1(F2) as the frequency (f) in Table 2; move the line to the next peak and record its frequency (3f); and measure the next three peaks and record.
- Measure the amplitudes of the harmonic peaks: press softkey FFT Menu; set Window to FlatTop (use FlatTop to measure amplitudes of peaks); press Cursor; press softkey Active Cursor, set f1 to On; move the vertical dashed line to the the fundamental peak; press Move f1 to Center; press softkey Active Cursor, set V1 to On; move the horizontal dashed line to the 0 Hz peak; record V1(F2) as the amplitude in dB; move the horizontal line to each peak and record the amplitude in Table 2. Repeat for the next five peaks.
- Since the displayed amplitudes are $\text{dB} = 20 \log(V_{\text{rms}}/1\text{v})$, compute the amplitudes from:
$$V_{\text{peak}} = [\text{antilog}(\text{dB}/20)] / (0.707)$$
. Record in Table 2.
- Compare theoretical and experimental values for frequencies and coefficients and record % differences.
- Optional: Repeat the above procedure for a sawtooth wave. Determine the experimental coefficients and harmonic frequencies. Different harmonics may be at work.



Questions:

1. What would be the value of a_0 if the wave of figure 1 varied from $-V_0$ to $+V_0$ instead of 0 to V_0 ?
2. What is the value of the voltage of the wave of figure 1 when $t = 1$ second?
3. Derive the coefficients for a sawtooth wave.

Derivation:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} V(t)d(\omega t) = \frac{1}{2\pi} [V_0 - 0] = \frac{V_0}{2}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} V(t)\cos(n\omega t) d(\omega t) = \frac{1}{\pi} \left[\frac{-V_0 \sin(n\omega t)}{n} \right]_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} V(t)\sin(n\omega t) d(\omega t) = \frac{1}{\pi} \left[\frac{-V_0 \cos(n\omega t)}{n} \right]_0^{2\pi} = \frac{V_0}{n\pi} [1 - \cos(n\pi)]$$

when $n = \text{odd}$, then $1 - \cos(n\pi) = 2$; when $n = \text{even}$, then $1 - \cos(n\pi) = 0$

Therefore, $V(t) = \frac{V_0}{2} + \frac{2V_0}{\pi} \left[\sin(\omega t) + \frac{1}{3} \sin(3\omega t) + \frac{1}{5} \sin(5\omega t) + \dots + \frac{1}{n} \sin(n\omega t) + \dots \right]$